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The Radon–Nikodým Property in Ordered Banach Spaces

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1. INTRODUCTION

Let X be an ordered Banach space. In this paper the following problem is studied: If the positive cone X_+ of X is almost generating (i.e., the space $X_+ - X_+$ is dense in X) and X_+ has the Radon–Nikodým property (RNP) does X have the RNP?

In KB-spaces the answer is positive (see [6, 3]). In [9; 11, p. 49] it is studied whether the sum of two closed, convex, and bounded sets with the RNP has the RNP. Using [11] we can show that if the positive cone X_+^* of X^* is generating ($X^* = X_+^* - X_+^*$) and X_+^* has the RNP then X^* has the RNP (Proposition 2.8).

In this paper it is shown that if the positive cone of $L_p(\mu, X)^*$ is almost generating and X_+^* has the RNP then the spaces X^* and $L_p(\mu, X)^*$ have the RNP. It is also proved that the cone X_+^* has the RNP if and only if the positive cones of $L_p(\mu, X)^*$ and $L_q(\mu, X^*)$ coincide. Finally, by an example it is shown that the answer to our problem is not positive in general.

For notation and terminology on vector measures we refer to [5]; on the RNP in subsets of X, to [2, 11]; and on ordered spaces, to [1, 7].

2. THE RADON–NIKODÝM PROPERTY IN X_+

Let X be a Banach space, C a wedge of X (i.e., $C \subseteq X$ with $kC + \lambda C = C$ for each $k, \lambda \in \mathbb{R}_+$) and (Ω, Σ, μ) a probability space. Suppose

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that X is an ordered space with positive wedge $X_+ = C$. The set $L_p^+(\mu, X) = \{f \in L_p(\mu, X) \mid f(t) \in X_+$, μ-almost everywhere$\}$, is the positive wedge of $L_p(\mu, X)$.

A closed, convex, and bounded subset K of X has the RNP with respect to the probability space (Ω, Σ, μ) if each μ-continuous vector measure $G: \Sigma \to X$ with $\mathrm{Ar}(G) = \{G(E)/\mu(E) \mid E \in \Sigma, \mu(E) \neq 0\} \subseteq K$, is representable; i.e., there exists $f \in L_1(\mu, X)$ such that $G(E) = \int_E f d\mu$, for each $E \in \Sigma$. The function f is called the Radon–Nikodým derivative of G with respect to μ and is denoted by $dG/d\mu$. Also $f(t) \in K$, μ-almost everywhere. The set $\mathrm{Ar}(G)$ is the average range of K with respect to μ.

Let D be a closed and convex subset of X. The set D has the RNP with respect to (Ω, Σ, μ) if each closed, convex, and bounded subset of D has the RNP with respect to (Ω, Σ, μ) or, equivalently, if each μ-continuous vector measure $G: \Sigma \to X$ with average range a bounded subset of D, is representable.

If D has the RNP with respect to any probability space (Ω, Σ, μ) then we say that the set D has the RNP.

If (Ω, Σ, μ) is purely atomic then D has the RNP with respect to (Ω, Σ, μ), because if the sequence (A_n) is an infinite partition of Ω consisting of atoms, then the function $f(t) = G(A_n)/\mu(A_n)$, $t \in A_n$, represents the vector measure G.

If (Ω, Σ, μ) is not purely atomic and D has the RNP with respect to (Ω, Σ, μ) then D has the RNP. This holds because the set D has the RNP with respect to (Ω', Σ', μ), where Ω' is the non-atomic part of Ω and $\Sigma' = \{E \cap \Omega' \mid E \in \Sigma\}$; therefore D has the RNP, by [2, Theorem 2.2.2, p. 20].

For a closed, convex subset D of X the following statements are equivalent:

(i) The set D has the RNP with respect to (Ω, Σ, μ).

(ii) Each μ-continuous vector measure $G: \Sigma \to X$ of bounded variation with $\mathrm{Ar}(G) \subseteq D$ is representable.

The proof is the following: We may assume that D contains the zero because a vector measure F is representable if and only if $G = F - x_0\mu$, where $x_0 \in X$ is representable. If $\mathrm{Ar}(G)$ is bounded then $|G|(\Omega) < +\infty$; therefore (ii) $\Rightarrow$ (i). Suppose that (i) holds and that G is a vector measure as in (ii). If $v = |G| + \mu$, then G is v-continuous and for each E with $\mu(E) > 0$, $G(E)/v(E) = (G(E)/\mu(E))(\mu(E)/v(E)) \in D$, because $G(E)/\mu(E) \in D$ and $\mu(E)/v(E) < 1$. Therefore the average range of G with respect to v is a bounded subset of D; hence dG/dv exists. Thus $(dG/dv) \cdot (dv/d\mu) = dG/d\mu$ and the statement (ii) is true.

In [8] an example of a μ-continuous vector measure G of bounded variation without derivative and values in the space c_0 of convergent to zero, real sequences are given. Since the values of G are positive, the positive cone c_0^+ of c_0 does not have the RNP.

In order to study the RNP in X_+ we show below that the Chatterji's pointwise convergence theorem and the main martingale convergence theorem hold also for D-values martingales, i.e., for martingales $(f_\tau, \Sigma_\tau)_{\tau\in T}$, such that for each τ, $f_\tau(t) \in D$, μ-almost everywhere. The proof is similar to the existing one and we do not present these results as new.

PROPOSITION 2.1. *Let D be a closed, convex subset of X. If D has the RNP with respect to (Ω, Σ, μ) then each D-valued, $L_1(\mu, X)$-bounded martingale $(f_n, \Sigma_n)_{n\in\mathbb{N}}$ converges pointwise almost everywhere.*

Proof. Assume that $\|f\|_1 \leq M$, for each n. Let ρ be a fixed point of $\mathbb{R}_+$. As in the proof of Chatterji's theorem given in [2, Theorem 2.2.7, p. 25], for each $t \in \Omega$ we put

$$\sigma(t) = \begin{cases} n, & \text{if } \|f_i(t)\| < \rho \text{ for each } i = 1, 2, \ldots, n-1 \text{ and } \|f_n(t)\| \geq \rho \\ +\infty, & \text{if } \|f_i(t)\| < \rho \text{ for each } i \in \mathbb{N}. \end{cases}$$

For each $n \in \mathbb{N}$ define the function $f_{\sigma\wedge n}\colon \Omega \to X$ by $f_{\sigma\wedge n}(t) = f_{\sigma(t)\wedge n}(t)$. Then

$$f_{\sigma\wedge n}(t) = f_1(t)X_{E_1}(t) + f_2(t)X_{E_2}(t) + \cdots + f_n(t)X_{E_n}(t),$$

where $E_i = \{t \in \Omega \mid \sigma(t) = i\}$ for $i = 1, 2, \ldots, n-1$, $E_n = \{t \in \Omega \mid \sigma(t) \geq n\}$, and X_{E_i} the characteristic function of E_i. Then $E_1, E_2, \ldots, E_n$ belong to Σ_n, $f_{\sigma\wedge n}$ has its values in D, and $f_{\sigma\wedge n} \in L_1(\mu, X)$.

As in [2], $(f_{\sigma\wedge n}, \Sigma_{\sigma\wedge n})$ is a martingale and

$$m(A) = \lim_{n\to\infty} \int_A f_{\sigma\wedge n}\, d\mu,$$

exists for each $A \in \Sigma$. Then m is a μ-continuous vector measure of bounded variation.

Since $f_{\sigma\wedge n}$ is D-valued,

$$\frac{1}{\mu(A)} \int_A f_{\sigma\wedge n}\, d\mu \in D,$$

and by the definition of m, $m(A)/\mu(A) \in D$. Therefore $\mathrm{Ar}(m) \subseteq D$ and $g = dm/d\mu$ exists, because D has the RNP. Following the proof of [2] we have that (f_n) converges almost everywhere.

PROPOSITION 2.2. *Let D be a closed and convex subset of X and let D have the RNP with respect to (Ω, Σ, μ). Suppose that $(f_\tau, \Sigma_\tau)_{\tau\in T}$ is a*

D-valued martingale in $L_p(\mu, X)$ and $1 \leq p < +\infty$. Then

(i) If $p = 1$, the net $(f_\tau)_{\tau \in T}$ converges in the $L_1(\mu, X)$-norm if and only if it is $L_1(\mu, X)$-bounded and uniformly integrable.

(ii) If $1 < p < +\infty$, the net $(f_\tau)_{\tau \in T}$ converges in the $L_p(\mu, X)$-norm if and only if it is $L_p(\mu, X)$-bounded.

Proof. (i) Suppose that $(f_\tau)_{\tau \in T}$ is $L_1(\mu, X)$-bounded and uniformly integrable. Without loss of generality we may assume that Σ is the σ-field generated by $B = U_{\tau \in T} \Sigma_\tau$. As in the proof of [5, V, Corollary 2.4, p. 127], denote by G the countably additive extension of the vector measure

$$F(E) = \lim_\tau \int_E f_\tau \, d\mu, \qquad E \in B,$$

on Σ. Then G is of bounded variation.

We shall show that $\mathrm{Ar}(G) \subseteq D$. For each $A \in \Sigma_\tau$ with $\mu(A) > 0$,

$$\frac{F(A)}{\mu(A)} = \frac{\int_A f_\tau \, d\mu}{\mu(A)} \in D,$$

because f_τ is D-valued. Let $E \in \Sigma$ with $\mu(E) > 0$. Since the metric space $B(\mu)$ associated with B is dense in $\Sigma(\mu)$, there exists a sequence (E_n) of B with $\mu(E_n \, \Delta E) \to 0$ and $F(E_n) \to G(E)$. So

$$\lim_n \frac{F(E_n)}{\mu(E_n)} = \frac{G(E)}{\mu(E)} \in D.$$

Therefore $\mathrm{Ar}(G) \subseteq D$ and the Radon–Nikodým derivative $f = dG/d\mu$ of G exists.

Then $\lim_\tau \int_E f_\tau \, d\mu = F(E) = G(E) = \int_E f \, d\mu$, $\forall \, E \in B$; therefore $\lim_\tau f_\tau = f$.

The proof of (ii) is the same as the corresponding proof of [5].

Denote by $L_p(\mu, D)$ the set:

$$L_p(\mu, D) = \{f \in L_p(\mu, X) \mid f(t) \in D, \ \mu\text{-almost everywhere}\}.$$

If D is closed and convex then $L_p(\mu, D)$ is also closed and convex.

PROPOSITION 2.3. *Let D be a closed and convex subset of X and $1 < p < +\infty$. Then the set D has the RNP if and only if $L_p(\mu, D)$ has the RNP.*

Proof. Let D have the RNP. Suppose that (S, F, λ) is a probability space, that $G : F \to L_p(\mu, X)$ is a μ-continuous vector measure, and that the average range of G with respect to λ is a bounded subset of $L_p(\mu, D)$. Then there exists $M \in \mathbb{R}_+$ such that $\|G(E)\| \leq M\lambda(E)$ for each $E \in F$. For each partition Π of S and Δ of Ω we define

$$f_{\Pi,\Delta}(s, w) = \sum_{E\in\Pi} \sum_{I\in\Delta} \frac{\int_I G(E)\, d\mu}{\lambda(E)\mu(I)} X_I(\omega)X_E(s).$$

If $\Sigma_{\Pi,\Delta}$ is the σ-field generated by $\Pi \times \Delta$, as in the proof of [5, V, Theorem 4.1, p. 140], $(f_{\Pi,\Delta}, \Sigma_{\Pi,\Delta})$ is a $L_p(S \times \Omega, F \times \Sigma, \lambda \times \mu, X)$-bounded martingale.

It is easy to show that this martingale is D-valued; therefore by the previous proposition, $f_{\Pi,\Delta}$ converges to a function $f \in L_p(\lambda \times \mu, X)$. As in [5] we define the function $g: S \to L_p(\mu, X)$ such that $g(s) = f(s, \cdot)$ and we show that

$$G(A) = \int_A g d\mu \qquad \text{for each } A \in F.$$

So the let $L_p(\mu, D)$ has the RNP.

For the converse we remark that the set $T(D) = \{T(x) = xX_\Omega \mid x \in D\}$, where X_Ω is the characteristic function of Ω, and has the RNP. Therefore the set D has the RNP by [11, p. 27].

Denote by $L(L_1(\mu), X)$ the set of linear, continuous operators of $L_1(\mu)$ into X and by $L^+(L_1(\mu), X)$ the set of positive ones. Let S_+ be the positive part of the unit sphere of $L_1(\mu)$, i.e.,

$$S_+ = \{h \in L_1^+(\mu) |\ \|h\| = 1\}.$$

By [11, p. 12], a closed and convex subset D of X has the RNP with respect to (Ω, Σ, μ) if and only if each $T \in L(L_1(\mu), X)$ with $T(S_+) \subseteq D$ is representable; i.e., there exists $f \in L_\infty(\mu, X)$ such that $T(h) = \int_\Omega fh\, d\mu$ for each $h \in L_1(\mu)$.

The operator T is positive if and only if $T(S_+) \subseteq X_+$, therefore in the case of X_+, the previous result has the form:

PROPOSITION 2.4. *Let X_+ be closed. Then X_+ has the RNP with respect to (Ω, Σ, μ) if and only if each positive, continuous linear operator $T: L_1(\mu) \to X$ is representable.*

Since the difference of two representable operators is representable, we have also the following.

PROPOSITION 2.5. *Suppose that each operator $T: L_1(\mu) \to X$ is regular (i.e., T is the difference of two positive ones). If X_+ has the RNP then X has the RNP.*

Remark 2.6. The assumption that each operator $T \in L(L_1(\mu), X)$ is regular implies that X_+ is generating as follows:

For each $x \in X$ the operator $T(h) = \langle l, h\rangle x$, $l \in L_\infty(\mu)$ is the difference of two positive operators T_1, T_2. If $h_0 \in L_1^+(\mu)$ with $\langle l, h_0\rangle = 1$, then

$$x = T(h_0) = T_1(h_0) - T_2(h_0) \in X_+ - X_+.$$

PROPOSITION 2.7. *If X is a Banach lattice and X_+ has the RNP then X has the RNP.*

Proof. Since c_0^+ does not have the RNP, X does not contain c_0 as a sublattice; therefore X is a KB-space. By [1, Theorem 15.3, p. 249], each operator $T \in L(L_1(\mu), X)$ is regular; therefore X has the RNP.

The following is an application of the main result of [9].

PROPOSITION 2.8. *Let τ be a Hausdorff vector space topology on X, coarser than the norm topology. Suppose that X_+ is generating, τ-closed and the positive part $U_+ = U \cap X_+$ of the closed unit ball U of X is τ-compact. If X_+ has the RNP, then X has the RNP.*

Proof. By [7, Theorem 3.5.2, p. 106], X_+ gives an open decomposition of X, so there exists $\rho \in \mathbb{R}$, such that $U_+ - U_+ \supseteq \rho U$. By our assumptions U_+ has the RNP and by [9, Proposition 1.6], the set $U_+ - U_+$ has the RNP. So U and, therefore, also the space X have the RNP.

Suppose that the functional $x^* \in X^*$ is strictly positive; i.e., $x^*(x) > 0$ for each $x \in X_+$ with $x \neq 0$. Then X_+ is a cone of X and the set $B = \{x \in X_+ \mid x^*(x) = 1\}$ is the base for X_+ defined by x^*.

PROPOSITION 2.9. *Let X_+ be a closed cone of X and let B be the base for X_+ defined by the strictly positive, continuous linear functional x^* of X. If the base B has the RNP, then the cone X_+ has the RNP.*

Proof. Let $F:\Sigma \to X$ be a μ-continuous vector measure of bounded variation with $\mathrm{Ar}(F) \subseteq X_+$. Let $b = x^*(F(\Omega))$. Then $\lambda(E) = (1/b)x^*(F(E))$, $E \in \Sigma$, is a μ-continuous probability measure and $G(E) = (1/b)F(E)$, $E \in \Sigma$, is a countably additive vector measure of bounded variation. If $\lambda(E) = 0$, then $F(E) = 0$ because x^* is strictly positive and $F(E) \in X_+$; therefore $G(E) = 0$. Hence, G is λ-continuous. Also $G(E)/\lambda(E) \in B$ for each E with $\lambda(E) > 0$, so $g = dG/d\lambda$ exists. If $h = d\lambda/d\mu$, then $F(E) = b\int_E hg\, d\mu$, for each E; therefore X_+ has the RNP.

Let the wedge C be closed. We say that C is isomorphic (or locally isomorphic) to a closed wedge Q of a Banach space Y, if there exists an additive, positive homogeneous, one-to-one linear map T of C onto Q and T, T^{-1} are continuous.

If the map T has a continuous, linear extension on the closed linear subspace $\overline{C - C}$ generated by C, then by [11, p. 27], C has the RNP if and only if Q has the RNP.

PROPOSITION 2.10. *Let X_+ be isomorphic to a closed wedge Q of a Banach space Y. Then X_+ has the RNP if and only if Q has the RNP.*

Proof. Suppose that $T: X_+ \to Q$ is an isomorphism of X_+ onto Q and Q has the RNP. By the continuity of T, T^{-1} on zero, there exist M, $N \in \mathbb{R}_+$ such that

$$N\|x\| \leq \|T(x)\| \leq M\|x\| \qquad \text{for each } x \in X_+.$$

We shall show that each closed, convex, and bounded subset K of X_+ is dentable [2, Theorem 2.3.6, p. 31]. Let $W = T(K)$. Then W is closed, convex, and bounded. Suppose that y_0 is a point of W, strongly exposed by $y_0^* \in Y^*$. The function $l(x) = y_0^*(T(x))$, $x \in X_+$, is continuous, additive, positive homogeneous, and exposes the point $x_0 = T^{-1}(y_0)$ in K (i.e., $l(x_0) > l(x)$, for each $x \in K$ with $x \neq x_0$).

For each sequence (x_v) of K with $l(x_v) \to l(x_0)$ we have

$$y_0^*(T(x_v)) \to y_0^*(y_0) \Rightarrow T(x_v) \to T(x_0) \Rightarrow x_v \to x_0.$$

So for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in K$, $\|x - x_0\| > \varepsilon$ implies that $l(x) < l(x_0) - \delta$.

Since l is additive, positive homogeneous, and continuous we have that x_0 does not belong to the closed convex hull of the set $\{x \in K \mid \|x - x_0\| \geq \varepsilon\}$; therefore K is dentable.

3. THE RADON–NIKODÝM PROPERTY IN ORDERED-DUAL SPACES

The topological dual X^* of X is an ordered space with positive wedge $X_+^* = \{x^* \in X^* \mid x^*(x) \geq 0 \text{ for each } x \in X_+\}$.

For each $1 \leq p < +\infty$ and $1 < q \leq +\infty$ with $1/p + 1/q = 1$, $L_q(\mu, X^*)$ is a closed subspace of $L_p(\mu, X)^*$; therefore the positive wedge $L_q^+(\mu, X^*) = \{f \in L_q(\mu, X^*) \mid f(t) \in X_+^*\ \mu\text{-almost everywhere}\}$ of $L_q(\mu, X^*)$ is a closed subset of the positive wedge $L_p(\mu, X)_+^* = \{l \in L_p(\mu, X)^* \mid l(f) \geq 0, \text{ for each } f \in L_p^+(\mu, X)\}$ of $L_p(\mu, X)^*$.

THEOREM 3.1. *For each $1 \leq p < +\infty$ and $1 < q \leq +\infty$ with $1/p + 1/p = 1$, the following statements are equivalent:*

(i) X_+^* has the RNP with respect to (Ω, Σ, μ).

(ii) $L_q^+(\mu, X^*) = L_p(\mu, X)_+^*$.

Proof. Let $L_q^+(\mu, X^*) = L_p(\mu, X)_+^*$. Assume that $G:\Sigma \to X^*$ is a μ-continuous vector measure with $\mathrm{Ar}(G)$ a bounded subset of X_+^*. Then there exists $M \in \mathbb{R}_+$ such that $\|G(E)/\mu(E)\| \leq M$ for each E with $\mu(E) > 0$.

Define l on the space of simple functions by

$$l\left(\sum_{i=1}^{n} x_i X_{E_i}\right) = \sum_{i=1}^{n} G(E_i)(x_i),$$

where X_{E_i} is the characteristic function of E_i and $x_i \in X$.

For each simple function f,

$$|l(f)| \leq \sum_{i=1}^{n} \left|\frac{G(E_i)}{\mu(E_i)}(\mu(E_i)x_i)\right| \leq M \sum_{i=1}^{n} \mu(E_i)\|x_i\| = M\|f\|_1 \leq M\|f\|_p;$$

therefore l is a linear, continuous functional of $L_p(\mu, X)$.

We shall show that l is positive; $l(f) \geq 0$ for each positive simple function $f = \sum_{i=1}^{n} x_i X_{E_i}$ because $x_i \in X_+$ and $G(E_i) \in X_+^*$. We shall show that the set of positive simple functions is dense in $L_p^+(\mu, X)$; therefore l is positive.

Let $h \in L_p^+(\mu, X)$. For each $\varepsilon > 0$ there exists a simple function

$$f = \sum_{i=1}^{n} x_i X_{E_i} \qquad \text{with } \|h - f\|_p < \partial, \qquad \text{where } 0 < \partial\,(1 + 2^{1/p}) < \varepsilon.$$

If $d_i = \inf\{\|h(t) - x_i\|^p | t \in E_i\}$ then

$$\sum_{i=1}^{n} \int_{E_i} d_i \, d\mu \leq \|h - f\|_p^p < \partial^p.$$

Let $t_i \in E_i$ with $\|h(t_i) - x_i\|^p < d_i + \partial^p$ and $w = \sum_{i=1}^{n} h(t_i) X_{E_i}$. Then $\|w - f\|_p^p = \sum_{i=1}^{n} \int_{E_i} \|h(t_i) - x_i\|^p \, d\mu \leq \sum_{i=1}^{n} \int_{E_i} (d_i + \partial^p) \, d\mu < 2\partial^p$. So $\|h - w\|_p \leq \|h - f\|_p + \|f - w\|_p < \partial + 2^{1/p}\,\partial < \varepsilon$, where w is positive.

By the assumption that $L_p(\mu, X)_+^* = L_q^+(\mu, X^*)$, there exists $g \in L_q^+(\mu, X^*)$ which represents l, i.e.,

$$l(f) = \int_\Omega \langle f, g\rangle \, d\mu \qquad \text{for each } \; f \in L_p(\mu, X).$$

Then

$$G(E)(x) = l(xX_E) = \int_\Omega \langle xX_E, g\rangle \, d\mu = \left(\int_E g \, d\mu\right)(x),$$

for each $E \in \Sigma$ and $x \in X$; therefore

$$G(E) = \int_E g \, d\mu \qquad \text{for each } E \in \Sigma.$$

So X_+^* has the RNP.

For the converse, suppose that X_+^* has the RNP with respect to (Ω, Σ, μ). Let $l \in L_p(\mu, X)_+^*$. For each $E \in \Sigma$ define $G(E)$ by

$$G(E)(x) = l(xX_E) \qquad \text{for each } x \in X.$$

Then $G(E)$ is a linear functional of X, $G(E)(x) = l(xX_E) \geq 0$ for each $x \in X_+$ and $\|G(E)(x)\| \leq \|l\| \, \|xX_E\|_p \leq \|x\| \, \|l\|\mu(E)$. So $G(E) \in X_+^*$ and $\|G(E)\| \leq \|l\|\mu(E)$.

The map $G : \Sigma \to X^*$ is a μ-continuous vector measure with $\mathrm{Ar}(G)$ a bounded subset of X_+^*. So there exists $g \in L_1^+(\mu, X^*)$ with

$$G(E) = \int_E g \, d\mu \quad \text{for each } E \in \Sigma.$$

Then

$$l(xX_E) = G(E)(x) = \int_\Omega \langle xX_E, g\rangle \, d\mu \qquad \text{for each } x \in X;$$

therefore

$$l(f) = \int_\Omega \langle f, g\rangle \, d\mu \qquad \text{for each simple function } f \in L_p(\mu, X).$$

Let $\Omega = \bigcup_{n=1}^{\infty} E_n \subseteq E_{n+1}$, such that $X_{E_n} g$ is bounded for each n.

Define

$$\begin{aligned} l_n(f) &= \int_\Omega \langle f, X_{E_n} g\rangle \, d\mu \\ &= \int_\Omega \langle X_{E_n} f, g\rangle \, d\mu, \qquad \text{for each } f \in L_p(\mu, X). \end{aligned}$$

Then l_n is a continuous linear functional which agrees with l on the set of simple functions supported by E_n. So

$$l_n(f) = l(X_{E_n} f) \qquad \text{for each } f \in L_p(\mu, X).$$

Since $X_{E_n} g$ represents l_n, $\|X_{E_n} g\|_q = \|l_n\| \leq \|l\|$; therefore $g \in L_q(\mu, X^*)$ by the monotone convergence theorem. But $\langle X_{E_n} f, g\rangle$, $\langle f, g\rangle \in L_1(\mu)$

by the Hölder inequality; therefore, $l(f) = \lim_{n\to+\infty} \int_\Omega \langle X_{E_n} f, g\rangle \, d\mu = \int_\Omega \langle f, g\rangle \, d\mu$. Thus g represents l; therefore $L_q^+(\mu, X^*) = L_p(\mu, X)_+^*$.

COROLLARY 3.2. *Let $1 \le p < +\infty$ and let the cone $L_p(\mu, X)_+^*$ be almost generating in $L_p(\mu, X)^*$. If X_+^* has the RNP, then the spaces X^* and $L_p(\mu, X)^*$ have the RNP.*

Proof. Suppose that X_+^* has the RNP. By our assumption that $L_p(\mu, X)_+^*$ is almost generating and the previous theorem, we have

$$\begin{aligned} L_p(\mu, X)^* &= \overline{L_p(\mu, X)_+^* - L_p(\mu, X)_+^*} \\ &= \overline{L_q^+(\mu, X^*) - L_q^+(\mu, X^*)_+} \subseteq L_q(\mu, X^*). \end{aligned}$$

Therefore $L_p(\mu, X)^* = L_q(\mu, X^*)$; hence X^* has the RNP by [5, IV, Theorem 3.1, p. 98]. Thus the space $L_q(\mu, X^*)$ and, therefore, also the space $L_p(\mu, X)^*$, have the RNP.

Remark 3.3. Suppose that $1 \le p < +\infty$. Then we have

(i) The wedge X_+^* is generating in X^* if and only if $L_p(\mu, X)_+^*$ is generating in $L_p(\mu, X)^*$.

(ii) If $L_p(\mu, X)_+^*$ is almost generating in $L_p(\mu, X)^*$, then X_+^* is almost generating in X^*.

To show (i) we suppose that X_+^* is generating. Then X_+ is a cone of X and by [7, 3.4.8, p. 102] X_+ is normal (or self-allied); i.e., there exists $a \in \mathbb{R}_+$, $a \neq 0$ such that for each $x, y \in X$, $0 \le x \le y$, implies $\|x\| \le a\|y\|$. It is easy to show that $L_p^+(\mu, X)$ is also a normal cone; therefore $L_p(\mu, X)_+^*$ is generating.

For the converse suppose that $x^* \in X^*$ and $\omega = x^* X_\Omega$. Then there exist $\omega_1, \omega_2 \in L_p(\mu, X)_+^*$ such that $\omega = \omega_1 - \omega_2$. Define x_1^*, x_2^* by $x_i^*(x) = \omega_i(xX_\Omega)$ for each $x \in X$ and $i = 1, 2$. Then $x_i^* \in X_+^*$ and

$$x^*(x) = \langle \omega, xX_\Omega\rangle = \omega_1(xX_\Omega) - \omega_2(xX_\Omega) = x_1^*(x) - x_2^*(x)$$

for each $x \in X$; therefore X_+^* is generating.

In the case where $L_p(\mu, X)_+^*$ is almost generating, we suppose that $\omega_1, \omega_2 \in L_p(\mu, X)_+^*$ with $\|\omega - (\omega_1 - \omega_2)\| < \varepsilon$. Then

$$|(x^* - (x_1^* - x_2^*))(x)| = |(\omega - (\omega_1 - \omega_2))(xX_\Omega)| < \varepsilon \, \|x\|$$

for each $x \in X$; therefore $\|x^* - (x_1^* - x_2^*)\| < \varepsilon$. So X_+^* is almost generating.

We give an example of an almost generating cone of c_0 with the RNP. Since c_0 does not have the RNP the answer to our problem is not positive, in general.

EXAMPLE 3.5. Let $X = c_0$, let C be the set of decreasing real sequences convergent to zero, let (e_n) be the usual basis of X, and let $b_n = \sum_{i=1}^{n} e_i$. Then (b_n) is a basis of X because for each $x = (x(i)) \in X$,

$$\sum_{i=1}^{n} (x(i) - x(i+1))b_i = \sum_{i=1}^{n} x(i)e_i - x(n+1)b_n$$

and $\lim_{n\to\infty} x(n+1)b_n = 0$. The set C is the positive cone of this basis, i.e., $C = \{\sum_{i=1}^{\infty} \lambda_i b_i \in X \mid \lambda_i \in \mathbb{R}_+ \text{ for each } i\}$. Therefore $\overline{C - C} = X$, because each finite linear combination of b_n is the difference of two positive ones.

The basis (b_n) is of type l_+; i.e., it is bounded and there exists $k \in \mathbb{R}_+$ such that $\|\sum_{i=1}^{n} a_i b_i\| \geq k \sum_{i=1}^{n} a_i$ for each finite sequence $a_1, a_2, \ldots, a_n$ of positive real numbers. By [10, II, Theorem 10.2, p. 323], the cone C is isomorphic (locally isomorphic) to the positive cone of l_1; therefore C has the RNP by Proposition 2.10.

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